

The Coin Exchange Problem and the Structure of Cube Tilings

Andrzej P. Kisielewicz and Krzysztof Przesławski

Wydział Matematyki, Informatyki i Ekonometrii, Uniwersytet Zielonogórski
ul. Z. Szafrana 4a, 65-516 Zielona Góra, Poland
A.Kisielewicz@wmie.uz.zgora.pl
K.Przeslawski@wmie.uz.zgora.pl

Abstract

Let k_1, \dots, k_d be positive integers and let D be a subset of $[k_1] \times \dots \times [k_d]$, whose complement can be decomposed into disjoint sets of the form $\{x_1\} \times \dots \times \{x_{s-1}\} \times [k_s] \times \{x_{s+1}\} \times \dots \times \{x_d\}$. We conjecture that the number of elements of D can be represented as a linear combination of the numbers k_1, \dots, k_d with non-negative integer coefficients. A connexion of this conjecture with the structure of periodical cube tilings is revealed.

For any positive integer n , we denote by $[n]$ the set $\{1, \dots, n\}$. We extend this notation to vectors $\mathbf{k} = (k_1, \dots, k_d)$ with positive integer coordinates: $[\mathbf{k}] := [k_1] \times \dots \times [k_d]$. If all k_i are greater than 1, then $[\mathbf{k}]$ is said to be a (*discrete*) *d*-box. A *line* in $[\mathbf{k}]$ is any set of the form

$$\{x_1\} \times \dots \times \{x_{s-1}\} \times [k_s] \times \{x_{s+1}\} \times \dots \times \{x_d\},$$

where $s \in [d]$, and $x_i \in [k_i]$. A subset D of $[\mathbf{k}]$ is said to be *complementable by lines* if its complement $[\mathbf{k}] \setminus D$ can be represented as a union of disjoint lines.

A non-negative integer n is *representable* by \mathbf{k} if there are non-negative integers n_1, \dots, n_d such that

$$n = n_1 k_1 + \dots + n_d k_d.$$

In other words, the amount n can be changed using coins of denominations k_1, \dots, k_d . As a consequence of this interpretation, the problem of representability is often called *the coin exchange problem*.

The following conjecture arises from certain problems concerning periodical cube tilings, as we shall explain it later on.

Conjecture

For each *d*-box $[\mathbf{k}]$, if $D \subseteq [\mathbf{k}]$ is complementable by lines, then the size $|D|$ of D is representable by \mathbf{k} .

It is not difficult to confirm this conjecture for $\mathbf{k} = (m, \dots, m, n)$, where m and n are arbitrary positive integers. If $d = 3$, then verification of the conjecture reduces to a strictly numerical problem:

Show that for every positive integers $1 < k_1 < k_2 < k_3$, and $1 \leq l_i \leq k_i - 1$, $i = 1, 2, 3$, the number

$$l_1 l_2 l_3 + (k_1 - l_1)(k_2 - l_2)(k_3 - l_3)$$

is representable by (k_1, k_2, k_3) .

This problem has been tested for a wide range of data by M. Hałuszczak and, independently, by A. Zieliński, an MSc student of the second author. In particular, it has been tested for all $1 < k_1 < k_2 < k_3 \leq 700$ and all l_i , $i = 1, 2, 3$, satisfying the constraints.

We define a *cube* in the d -dimensional Euclidean space \mathbb{R}^d to be any translate of the unit cube $[0, 1)^d$. Let T be a subset of \mathbb{R}^d . The family $[0, 1)^d + T := \{[0, 1)^d + t : t \in T\}$ is said to be a *cube tiling* of \mathbb{R}^d if for each pair of distinct vectors $s, t \in T$ the cubes $[0, 1)^d + s$ and $[0, 1)^d + t$ are disjoint and $\bigcup [0, 1)^d + T = \mathbb{R}^d$. Let $\mathbf{k} := (k_1, \dots, k_d)$ be a vector with all coordinates that are positive integers. The tiling $[0, 1)^d + T$ is said to be *\mathbf{k} -periodic* if for every vector of the standard basis $e_1 = (1, 0, \dots, 0), \dots, e_d = (0, \dots, 1)$ one has

$$T + k_i e_i = T.$$

We define the (*flat*) *torus* $\mathbb{T}_{\mathbf{k}}^d$, to be the set $[0, k_1) \times \dots \times [0, k_d)$ with addition mod \mathbf{k} :

$$x \oplus y := ((x_1 + y_1) \bmod k_1, \dots, (x_d + y_d) \bmod k_d).$$

We can extend the notion of a cube so that it will apply to flat tori: Cubes in $\mathbb{T}_{\mathbf{k}}^d$ are the sets of the form $[0, 1)^d \oplus t$, where $t \in \mathbb{T}_{\mathbf{k}}^d$. It is clear that we can speak about cube tilings of $\mathbb{T}_{\mathbf{k}}^d$ and that there is a canonical ‘one-to-one’ correspondence between these tilings and the \mathbf{k} -periodic tilings of \mathbb{R}^d .

From now on, \mathbf{k} is assumed to have all coordinates greater than 1.

If $[0, 1)^d \oplus T$ is a cube tiling of $\mathbb{T}_{\mathbf{k}}^d$ and $S \subseteq T$, then we say that the packing $[0, 1)^d \oplus S$ is a *simple component* of the cube tiling if S is an equivalence class of the relation ‘ \sim ’ defined on T as follows:

$$x \sim y \text{ if and only if } x - y \in \mathbb{Z}^d.$$

For each $t \in \mathbb{T}_{\mathbf{k}}^d$, the *integer code* of t is defined by

$$\varepsilon(t) := (\lfloor t_1 \rfloor + 1, \dots, \lfloor t_d \rfloor + 1).$$

Clearly, ε maps $\mathbb{T}_{\mathbf{k}}^d$ into $[\mathbf{k}]$.

One can prove the following rather non-trivial result.

THEOREM 1 *If $[0, 1)^d + S$ is a simple component of a cube tiling $[0, 1)^d \oplus T$ of $\mathbb{T}_{\mathbf{k}}^d$, then $\varepsilon(S) \subseteq [\mathbf{k}]$ is complementable by lines.*

If $D \subseteq [\mathbf{k}]$ is complementable by lines, then there is a cube tiling $[0, 1)^d \oplus T$ of $\mathbb{T}_{\mathbf{k}}^d$ with a simple component $[0, 1)^d + S$ such that $\varepsilon(S) = D$.

If, in addition, we take into account that by Keller's theorem (see any of the three papers we refer to), the restriction $\varepsilon|_T$ of ε to T is a bijection for each cube tiling $[0, 1)^d \oplus T$, then the conjecture can be rephrased as follows:

If $[0, 1)^d + S$ is a simple component of the cube tiling $[0, 1)^d + T$ of the torus $\mathbb{T}_{\mathbf{k}}^d$, then the size $|S|$ of S is representable by \mathbf{k} .

References

- [1] A. IOSEVICH, S. PEDERSEN, Spectral and Tiling Properties of the Unit Cube, *Inter. Math. Res. Notices* **16** (1998), 819–828.
- [2] O.-H. KELLER, Über die lückenlose Erfüllung des Raumes mit Würfeln, *J. Reine Angew. Math.* **163** (1930), 231–248.
- [3] O. PERRON, Über lückenlose Ausfüllung des n -dimensionalen Raumes durch kongruente Würfel, *Math. Z.* **46** (1940), 1–26.